

A Level Further Mathematics B (MEI) Y434 Numerical Methods Sample Question Paper

Date – Morning/Afternoon

Time allowed: 1 hour 15 minutes

OCR supplied materials:

- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You must have:

- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)
- Scientific or graphical calculator

MODEL
ANSWERS



INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- In each question you must show sufficient detail of the method(s) which you are using.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION

- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets [].
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

1 (i) Solve the following simultaneous equations.

$$\begin{aligned}x + y &= 1 \\x + 0.99y &= 2\end{aligned}$$

[2]

(ii) The coefficient 0.99 is correct to two decimal places. All other coefficients in the equations are exact. With the aid of suitable calculations, explain why your answer to part (i) is unreliable. [3]

$$\begin{aligned}\text{i. } x + y &= 1 && \Rightarrow x = 1 - y \\x + 0.99y &= 2 && x = 2 - 0.99y\end{aligned}$$

$$1 - y = 2 - 0.99y$$

$$-1 = 0.01y$$

$$y = \underline{-100}$$

$$x = 1 - (-100)$$

$$x = \underline{101}$$

$$\text{ii. } y = \frac{-1}{1 - 0.99} \Rightarrow \frac{-1}{1 - 0.995} < y < \frac{-1}{1 - 0.985}$$

$$-200 < y < -66.67$$

$$x = 1 - y \Rightarrow 1 - (-66.67) < x < 1 - (-200)$$

$$67.67 < x < 201$$

because we are subtracting 2 numbers that are very similar, the range for x & y is huge

- 2 The following spreadsheet printout shows the bisection method being applied to the equation $f(x) = 0$, where $f(x) = e^x - x^2 - 2$. Some values of $f(x)$ are shown in columns B and D.

	A	B	C	D	E	F	G
1	a	$f(a)$	b	$f(b)$	$(a+b)/2$	$f((a+b)/2)$	mpe
2	1	-0.28172	2	1.389056	1.5	0.231689	0.5
3	1	-0.28172	1.5	0.231689	1.25	-0.072157	0.25
4	1.25	-0.07216	1.5	0.231689	1.375	0.064452	0.125
5	1.25	-0.07216	1.375	0.064452	1.3125	-0.007206	0.0625
6	1.3125	-0.00721	1.375	0.064452	1.34375	0.027728	0.03125

- (i) The formula in cell A3 is $=IF(F2>0, A2, E2)$. State the purpose of this formula. [1]
- (ii) The formula in cell C3 is $=IF(F2>0, \dots, \dots)$. What are the missing cell references? [1]
- (iii) In which row is the magnitude of the maximum possible error (mpe) less than 5×10^{-7} for the first time? [2]

i. check if I needs to replace 1.5 by checking the sign of F2

ii. $=IF(F2 > 0, E2, C2)$

iii. $0.5^{n-1} < 5 \times 10^{-7}$

$$0.5^n < 2.5 \times 10^{-7}$$

$$n=20: 0.5^{20} = 9.5 \times 10^{-7}$$

$$n=21: 0.5^{21} = 4.77 \times 10^{-7}$$

$$n=22: 0.5^{22} = 2.38 \times 10^{-7}$$

$$2.38 \times 10^{-7} < 2.5 \times 10^{-7}$$

so this happens on the 22nd row

3 The equation $\sinh x + x^2 - 1 = 0$ has a root, α , such that $0 < \alpha < 1$.

(i) Verify that the iteration $x_{r+1} = \frac{1 - \sinh x_r}{x_r}$ with $x_0 = 1$ fails to converge to this root. [2]

(ii) Use the relaxed iteration $x_{r+1} = (1 - \lambda)x_r + \lambda \left(\frac{1 - \sinh x_r}{x_r} \right)$ with $\lambda = \frac{1}{4}$ and $x_0 = 1$ to find α correct to 6 decimal places. [2]

$$\begin{aligned} \text{i. } x_{r+1} &= \frac{1 - \sinh x_r}{x_r} & x_0 &= 1 \\ & & x_1 &= -0.175 \\ & & x_2 &= -6.713\dots \\ & & x_3 &= -61.443 \end{aligned}$$

so this does not converge

$$\text{ii. } x_{r+1} = (1 - \lambda)x_r + \lambda \left(\frac{1 - \sinh x_r}{x_r} \right)$$

$$\lambda = \frac{1}{4}$$

$$x_0 = 1$$

$$x_1 = 0.706120$$

$$x_2 = 0.612353$$

$$x_3 = 0.601606$$

$$x_4 = 0.601403$$

$$x_5 = 0.601402$$

$$x_6 = 0.601402$$

$$\text{so } \alpha = 0.601402$$

- 4 The table below gives values of a function $y = f(x)$.

x	0.2	0.3	0.35	0.4	0.45	0.5	0.6
$f(x)$	0.789922	0.754628	0.749199	0.749997	0.756257	0.767523	0.804299

- (i) Calculate three estimates of $\frac{dy}{dx}$ at $x=0.4$ using the central difference method. [4]
- (ii) State the value of $\frac{dy}{dx}$ at $x=0.4$ to an appropriate degree of accuracy. Justify your answer. [2]

$$\begin{aligned}
 \text{i. } h=0.2: \quad \frac{dy}{dx} &\approx \frac{f(0.6) - f(0.2)}{0.6 - 0.2} \\
 &= \frac{0.804299 - 0.789922}{0.4} \\
 &= 0.0359425
 \end{aligned}$$

$$\begin{aligned}
 h=0.1: \quad \frac{dy}{dx} &\approx \frac{f(0.5) - f(0.3)}{0.5 - 0.3} \\
 &= \frac{0.767523 - 0.754628}{0.2} \\
 &= 0.064475
 \end{aligned}$$

$$\begin{aligned}
 h=0.05: \quad \frac{dy}{dx} &\approx \frac{f(0.45) - f(0.35)}{0.45 - 0.35} \\
 &= \frac{0.756257 - 0.749199}{0.1}
 \end{aligned}$$

$$= 0.07058$$

ii. 0.07 is reasonable

$h=0.05$ is the smallest h , so gives the most accurate estimate

- 5 A vehicle is moving in a straight line. Its velocity at different times is recorded and shown below. The velocities are recorded to 5 significant figures and the times may be assumed to be exact.

Time (t seconds)	5	10	12	15
Velocity (v metres per second)	5.1250	11.000	14.000	18.375

It is suggested initially that a quadratic model may be appropriate for this situation.

- (i) Given that the vehicle is modelled as a particle with constant mass, what assumption about the net force acting on the vehicle leads to a quadratic model? [1]
- (ii) Find Newton's interpolating polynomial of degree 2 to model this situation. Write your answer in the form $v = at^2 + bt + c$. [4]
- (iii) Comment on whether this model appears to be appropriate. [2]
- (iv) Use this model to find an approximation to the distance travelled over the interval $5 \leq t \leq 15$. [2]
- Further investigation suggests that a cubic model may be more appropriate.
- (v) What technique would you use to fit a cubic model to the data in the table? [1]

i. force is a linear function of time

ii. Use the values of $t = 5, 10, 15$

$$\begin{array}{r}
 5.1250 \\
 \\
 11 \\
 \\
 18.375
 \end{array}
 \begin{array}{l}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} 5.875 \\
 \left. \begin{array}{l} \\ \\ \end{array} \right\} 7.375 \\
 \left. \begin{array}{l} \\ \\ \end{array} \right\} 1.5
 \end{array}$$

1st difference 2nd difference

$$v = 5.125 + \frac{(t-5)}{5} \times (5.875) + \frac{(t-5)(t-10)}{5^2 \times 2!} \times 1.5$$

$$= 5.125 + 1.175t - 5.875 + \frac{3(t^2 - 15t + 50)}{100}$$

$$= 1.175t - 0.75 + 0.03t^2 - 0.45t + 1.5$$

$$V = 0.03t^2 + 0.725t + 0.75$$

$$\begin{aligned} \text{iii. } f(12) &= 0.03(12)^2 + 0.725(12) + 0.75 \\ &= 13.77 \end{aligned}$$

13.77 is close to 14.000, so the model seems to be appropriate

$$\begin{aligned} \text{iv. } S &= \int_5^{15} v \, dt = \int_5^{15} (0.03t^2 + 0.725t + 0.75) \, dt \\ &= \left[0.01t^3 + 0.3625t^2 + 0.75t \right]_5^{15} \\ &= 0.01(15)^3 + 0.3625(15)^2 + 0.75(15) \\ &\quad - 0.01(5)^3 - 0.3625(5) - 0.75(5) \\ &= 112.5 \end{aligned}$$

v. Lagrange's form of the interpolating polynomial

6 The secant method is to be used to solve the equation $x - \ln(\cos x) - 1 = 0$.

- (i) Show that starting with $x_0 = -1$ and $x_1 = 0$ leads to the method failing to find the root between $x = 0$ and $x = 1$. [3]

The spreadsheet printout shows the application of the secant method starting with $x_0 = 0$ and $x_1 = 1$. Successive approximations to the root are in column E.

	A	B	C	D	E
1	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$	x_{n+2}
2	0	-1	1	0.6156265	0.6189549
3	1	0.6156265	0.6189549	-0.175846	0.7036139
4	0.6189549	-0.1758461	0.7036139	-0.025245	0.7178053
5	0.7036139	-0.0252451	0.7178053	0.0011619	0.7171808
6	0.7178053	0.0011619	0.7171808	-7.4E-06	0.7171848
7	0.7171808	-7.402E-06	0.7171848	-2.16E-09	0.7171848
8	0.7171848	-2.16E-09	0.7171848	3.997E-15	0.7171848

- (ii) What feature of column B shows that this application of the secant method has been successful? [1]
- (iii) Write down a suitable spreadsheet formula to obtain the value in cell E2. [2]

Some analysis of convergence is carried out, and the following spreadsheet output is obtained.

	A	B	C	D	E	F	G	H
1	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$	x_{n+2}			
2	0	-1	1	0.6156265	0.6189549	0.084659	0.167629	1.980053
3	1	0.6156265	0.6189549	-0.175846	0.7036139	0.0141913	-0.044	-3.10054
4	0.6189549	-0.1758461	0.7036139	-0.025245	0.7178053	-0.0006244	-0.00633	10.13727
5	0.7036139	-0.0252451	0.7178053	0.0011619	0.7171808	3.953E-06	0.000292	73.83899
6	0.7178053	0.0011619	0.7171808	-7.4E-06	0.7171848	1.154E-09	-1.8E-06	
7	0.7171808	-7.402E-06	0.7171848	-2.16E-09	0.7171848	-2.109E-15		
8	0.7171848	-2.16E-09	0.7171848	3.997E-15	0.7171848			

The formula in cell F2 is $=E3 - E2$. The formula in cell G2 is $=F3/F2$. The formula in cell H2 is $=F3/(F2^2)$.

- (iv) (A) Explain the purpose of each of these three formulae. [3]
- (B) Explain the significance of the values in columns G and H in terms of the rate of convergence of the secant method. [2]
- (v) Explain why the values in cells F6 and F7 are not 0. [1]

$$i. f(x) = x - \ln \cos x - 1$$

$$x_0 = -1, x_1 = 0, x_2 = 0 - \frac{(0 - 0 - 1)(0 - (-1))}{(0 - 0 - 1) - (-1 - \ln 0.5403)}$$

$$= 2.601636$$

to find x_3 we need $\ln(\cos(2.60\dots))$

$$\cos(2.60\dots) = -0.8577$$

$\ln(-0.8577)$ is undefined, so this method fails

ii. the values are converging to zero

iii. $(A_2^* D_2 - C_2^* B_2) / (D_2 - B_2)$

iv. A) = $E_2 - E_1$ finds the difference between the first

2 estimates

= F_3 / F_2 finds the ratio of differences

for the first two estimates

= $F_3 / (F_2^2)$ finds the ratio of

differences of each estimate to the square of the previous

B) the values in column G suggest convergence

is faster than 1st order.

the values in H suggest the convergence is slower than 2nd order

v. the values in column E are stored to greater accuracy than displayed

7 Fig. 7 shows the graph of $y = f(x)$ for values of x from 0 to 1.

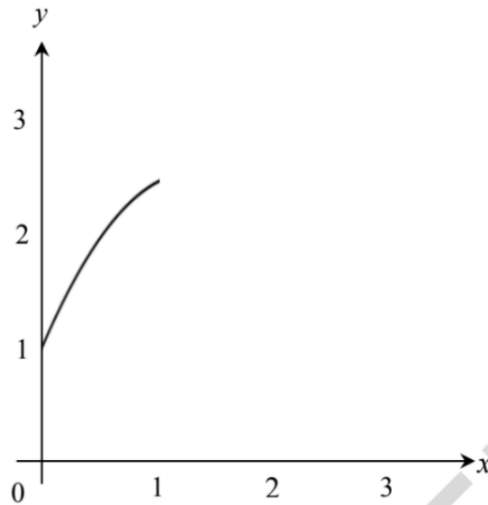


Fig. 7

The following spreadsheet printout shows estimates of $\int_0^1 f(x)dx$ found using the midpoint and trapezium rules for different values of h , the strip width.

	A	B	C
1	h	Midpoint	Trapezium
2	1	1.99851742	1.751283839
3	0.5	1.9638591	1.874900631
4	0.25	1.95135259	1.919379864
5	0.125	1.94682102	1.935366229

- (i) Without doing any further calculation, write down the smallest possible interval which contains the value of the integral. Justify your answer. [2]
- (ii) (A) • Calculate the ratio of differences, r , for the sequence of estimates calculated using the trapezium rule.
- Hence suggest a value for r correct to 2 significant figures.
 - Comment on your suggested value for r . [3]
- (B) • Use extrapolation to find an improved approximation to the value of the integral.
- State the value of the integral to two decimal places.
 - Explain why this precision is secure. [4]

$$i. 1.9353662285 \leq I < 1.946821025$$

the curve is concave, so curves upwards, meaning the trapezium rule provides an underestimate & the midpoint rule provides an overestimate

	differences	ratios
ii. A) 1.751283839	0.123616792	0.3598155
1.874900631		
1.919379864	0.044479233	0.3594119
1.935366229	0.015986368	

so $r \approx 0.36$

the trapezium rule is a 2nd order method, so r should be close to 0.25

$$\begin{aligned}
 \text{B) } & 1.93536623 + 0.01598637 \times \frac{0.36}{0.64} \\
 & = 1.9443586
 \end{aligned}$$

So the value of the integral is 1.94 to 2 d.p., which agrees with T_8 .

Using a similar approach with the sequence of estimates calculated using the midpoint rule, the approximation to the integral from extrapolation was found to be 1.944 27 correct to 5 decimal places.

(iii) Andrea uses the extrapolated midpoint rule value and the value found in part (ii) (B) to write down an interval which contains the value of the integral. Comment on the validity of Andrea's method. [2]

(iv) Use the values from the spreadsheet output to calculate an approximation to the integral using Simpson's rule with $h = 0.125$. Give your answer to 5 decimal places. [2]

Approximations to the integral using Simpson's rule are given in the spreadsheet output below. The number of applications of Simpson's rule is given in column N.

N	O	P	Q
n	Simpson	differences	ratio
1	1.91610623	0.01810005	0.3584931
2	1.93420628	0.00648874	0.3556525
4	1.94069502	0.00230774	0.3544828
8	1.94300275	0.00081805	0.3539885
16	1.94382081	0.00028958	0.3537638
32	1.94411039	0.00010244	0.3536568
64	1.94421283	3.623E-05	
128	1.94424906		

(v) Use the spreadsheet output to find the value of the integral as accurately as possible. Justify the precision quoted. [6]

END OF QUESTION PAPER

iii. extrapolation is only an approximate method, so Andrea's method is not appropriate

iv. When $h = 0.125$, $I \approx \frac{1.91937986 + 2 \times 1.95135259}{3}$
 $= 1.940695013$
 $= \underline{1.94070}$

$$\begin{aligned} \text{V. Extrapolation: } & 1.94424906 + 0.00003623 \times \frac{0.354}{1-0.354} \\ & = 1.944268914 \end{aligned}$$

SO we can say that it is 1.9443 to 4 d.p.
because this agrees with 1.94424906